A heralded two-qutrit entangled state

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Abstract. We propose a scheme for building a heralded two-qutrit entangled state from polarized photons. An optical circuit is presented to build the maximally entangled two-qutrit state from two heralded Bell pairs and ideal threshold detectors. Several schemes are discussed for constructing the two Bell pairs. We also show how one can produce an unbalanced two-qutrit state that could be of general purpose use in some protocols. In terms of applications of the maximally entangled qutrit state, we mainly focus on how to use the state to demonstrate a violation of the Collins-Gisin-Linden-Massar-Popescu inequality under the restriction of measurements which can be performed using linear optical elements and photon counting. Other possible applications of the state, such as for higher dimensional quantum cryptography, teleportation, and generation of heralded two-qudit states are also briefly discussed.

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1. Introduction

Sources of entangled photons are important ingredients for optical quantum information processing. A convenient way to generate maximally entangled photons is based on spontaneous parametric down-conversion (PDC) that has been used to demonstrate prototypes of many applications in quantum information processing [1, 2]. However, to satisfy scalability requirements for quantum computing and other protocols, deterministic or at least "heralded" (event ready) photon sources are required [3, 4].

The basic unit of quantum information is the qubit. However a variety of results [5, 6, 7, 8] have shown that multi-level systems - qu*d*its (d; dimensions) have advantages over qubits in certain circumstances. In particular, it is experimentally feasible to control several photons very precisely as such higher dimensional objects [9, 10, 11], and photonic qutrits have been implemented using PDC in orbital angular momentum, polarization, multi-mode, and energy-time [12, 13, 14, 15].

We analyze a scheme for building a heralded two-qutrit state from polarized photons. To achieve the qutrit state, generation of two heralded Bell pairs and four quantum memories are required. The Bell pairs are mixed at two beam-splitters (BSs), and two null detections herald the desired qutrit state in two spatial modes. The detector is assumed an ideal threshold detector, which does not need to be number resolving - it need only trigger on non-absorption versus absorption. We discuss how one can produce an unbalanced two-qutrit entangled state that may be of more general purpose use in some protocols [5]. In terms of applications of the qutrit state, we focus on how to use it to demonstrate a violation of the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality [16].

The paper is organized as follows. In Section 2, we introduce notation and also discuss how a typical state generated by Type-II PDC includes a two-qutrit state as a third-order process. In Section 3, we discuss several schemes to construct two heralded Bell pairs and show an optical circuit to build a heralded two-qutrit state from the Bell pairs. In Section 4 we discuss the ability of the single-qutrit state to violate the CGLMP inequality in the case that the measurements performed are done with linear optics (which restricts the possible qutrit operations from $U(3)$ to those of a 3-dimensional representation of $SU(2)$). Other possible applications of the qutrit state, such as for higher dimensional quantum cryptography, teleportation, and generation of heralded two-qudit states are also briefly discussed. In Section 5, we conclude by summarizing the results.

2. Background

In this section, we present our notation for maximally entangled states in polarization and a typical generation scheme of polarization entangled states using the PDC method that has been used for many important proof-of-principle demonstrations of quantum information processing. We show how from Type-II PDC photons, a two-qutrit

entangled state in polarization can be probabilistically observed by post-selection.

2.1. Maximally entangled states in polarization

Let us first define a qudit state with basis $|i\rangle$ $(i = 0, 1, ..., d-1)$. A maximally entangled bipartite state between A and B for qudits is represented by

$$|\psi^d\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A |i\rangle_B \in \mathcal{H}_d \otimes \mathcal{H}_d = d \otimes d. \tag{1}$$

$|i\rangle$ is an orthonormal basis of $\mathcal{H}_d$. One of the Bell pairs is represented by

$$|\psi^2\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B), \tag{2}$$

and

$$|\psi^3\rangle_{AB} = \frac{1}{\sqrt{3}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B + |2\rangle_A|2\rangle_B). \tag{3}$$

We denote a photonic state

$$|m, n\rangle_A = \frac{(a_H^\dagger)^m (a_V^\dagger)^n}{\sqrt{m!n!}} |0, 0\rangle_A \tag{4}$$

consisting of m horizontal photons and n vertical photons in spatial mode A ($a_{H(V)}^\dagger$; creation operator for horizontal (vertical) polarization, $|0, 0\rangle$; a vacuum state), the photonic state $|m, n\rangle_A$ spans $\mathcal{H}_d$ as an orthonormal basis. Thus, the state in Eq. (1) is equal to

$$|\psi^d\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |d-1-i, i\rangle_A |d-1-i, i\rangle_B. \tag{5}$$

For example, the Bell state is represented by

$$|\psi^2\rangle_{AB} = \frac{1}{\sqrt{2}}(|1, 0\rangle_A|1, 0\rangle_B + |0, 1\rangle_A|0, 1\rangle_B). \tag{6}$$

A maximally entangled state for qutrits is given by

$$|\psi^3\rangle_{AB} = \frac{1}{\sqrt{3}}(|2, 0\rangle_A|2, 0\rangle_B + |1, 1\rangle_A|1, 1\rangle_B + |0, 2\rangle_A|0, 2\rangle_B). \tag{7}$$

Therefore, three basis vectors for qutrits can be defined by $|0\rangle \equiv |2, 0\rangle$, $|1\rangle \equiv |1, 1\rangle$, and $|2\rangle \equiv |0, 2\rangle$.

2.2. Photon-pair source in PDC

Using Type-II PDC scheme, the superposition of the maximally entangled states can be generated. A PDC photonic state in modes A and B is equal to

$$|\Psi^{\mathrm{PDC}}\rangle_{AB} = \mathrm{e}^{-\mathrm{i}Ht}|0, 0\rangle = \frac{1}{\cosh^2\tau} \sum_{d=1}^{\infty} \sqrt{d} \tanh^{d-1}\tau |\psi_-^d\rangle_{AB}, \tag{8}$$

where

$$H = \mathrm{i}\kappa(a_H^\dagger b_V^\dagger - a_V^\dagger b_H^\dagger) + \mathrm{H.c.}, \tag{9}$$

and $\tau = \kappa t$ is the effective interaction time (H.c.; Hermitian conjugate) [17], and

$$|\psi_{-}^{d}\rangle_{\mathrm{AB}} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} (-1)^{i} |d-i-1, i\rangle_{A} |i, d-i-1\rangle_{B}, \tag{10}$$

where $|\psi_{-}^{1}\rangle_{AB} \equiv |\psi^{1}\rangle_{AB}$ is a vacuum state in modes A and B. For simplicity, we only consider the first three terms in Eq. (8), and the normalized state is

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{N}} \left(\alpha_1 |\psi_{-}^{1}\rangle_{AB} + \sqrt{2}\alpha_2 |\psi_{-}^{2}\rangle_{AB} + \sqrt{3}\alpha_3 |\psi_{-}^{3}\rangle_{AB} \right), \tag{11}$$

where

$$N = (\alpha_1)^2 + 2(\alpha_2)^2 + 3(\alpha_3)^2, \tag{12}$$

and

$$\alpha_d \equiv \tanh^{d-1}\tau / \cosh^2\tau. \tag{13}$$

When a polarization rotator $U(\pi/2)$ is applied in modes B with angle rotation $\pi/2$ (see Appendix A), the state is equal to

$$|\Psi'\rangle_{AB} = \left[\hat{I}_A \otimes U_B(\pi/2)\right] |\Psi\rangle_{AB}, \tag{14}$$

$$= \frac{1}{\sqrt{N}} \left(\alpha_1 |\psi^{1}\rangle_{AB} - \sqrt{2}\alpha_2 |\psi^{2}\rangle_{AB} + \sqrt{3}\alpha_3 |\psi^{3}\rangle_{AB} \right), \tag{15}$$

where $\hat{I}$ is the identity operator. The success probability to obtain the qutrit state $|\psi^{3}\rangle_{AB}$ is $3\alpha_3^2/N$. Experimentally [13], the detection of $|\psi^{3}\rangle_{AB}$ is performed post-selectively by sending the two photons through a PBS followed by two number resolving detections (themselves possible cascaded threshold detectors [24]).

3. Building a heralded two-qutrit entangled state

As PDC is a probabilistic emission process including vacuum and higher-order photons, a problem arises if we wish to use these photons in scalable approaches to practical quantum information processing. Since the photons are not heralded, all input photons must be measured to ensure that desired states are prepared - these are often used the schemes of post-selection [18]. We now consider an alternative to generate a heralded qutrit state encoded in the photon polarization (no longer a vacuum state and other higher-order photons). The qutrit state is constructed from two heralded Bell pairs and conditioned on a null detection performed by two ideal threshold detectors.

3.1. Building two heralded Bell pairs

Let us first assume that heralded Bell-pair generators (HBPGs) and quantum memories are prepared as shown in Fig. 1(a). The two heralded Bell pairs are emitted in modes A and B, and C and D. The HBPG has been studied both theoretically and experimentally in various physical systems [19, 4]. In fact an experimental realization has recently been achieved via a bi-exciton cascade in semiconductor quantum dots [19], while the

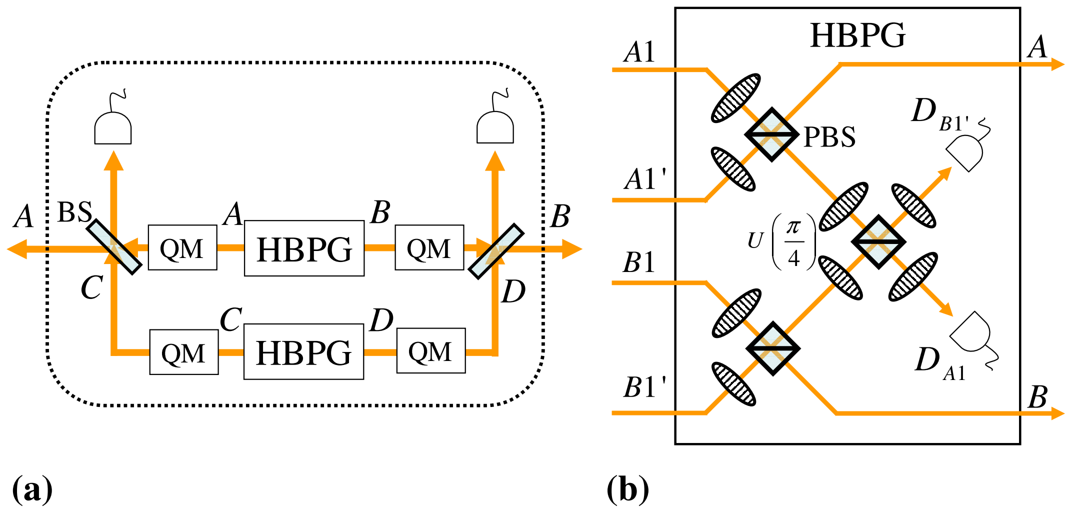

Figure 1. (a) A setup is depicted to build a heralded entangled two-qutrit state. This setup consists of two BSs, two ideal threshold detectors, two heralded Bell-pair generators (HBPGs) and four quantum memories (QMs). (b) A circuit of HBPG consists of three PBSs, eight polarization rotators, and two detectors [21]. If two out of four single photons are simultaneously detected in detectors D_{A1} and $D_{B1'}$ respectively, the outcome state is a Bell pair in modes A and B. (BS; beam-splitter, PBS; polarizing beam-splitter, $U(\pi/4)$; polarization rotator with angle $\pi/4$, and D_K; detector in mode K)

theoretical approach of Ref. [4] described how to probabilistically produce heralded two-photon entangled states using controlled-NOT operations on two PDC sources. Note that even if a heralded two-photon source is available, four quantum memories (e.g., atomic ensembles, cavity QED systems, optical loops [20]) will be required to ensure that the two heralded Bell pairs simultaneously arrive at the two BSs of Fig. 1(a).

Instead of the aforementioned HBPG schemes, heralded single photon sources can be used to generate heralded Bell pairs, as depicted in Fig. 1(b). This circuit creates a heralded Bell pair in modes A and B by destroying two out of four input single photons [21]. The photons enter with horizontal polarization and are rotated by the polarization rotator $U(\pi/4)$ that makes them an equal superposition of horizontal and vertical polarizations. The state emerging from each of the PBS is such that it is a superposition of the Bell states (see Eq. (6))

$$(|1,0\rangle_{A1}|1,0\rangle_{A1'} + |0,1\rangle_{A1}|0,1\rangle_{A1'}) \otimes (|1,0\rangle_{B1}|1,0\rangle_{B1'} + |0,1\rangle_{B1}|0,1\rangle_{B1'}), \quad (16)$$

and other terms that have the two photons in the same spatial mode, which cannot lead to acceptable detections in the end. A Type-II fusion gate in modes $A1$ and $B1'$ then fuses the two Bell pairs from the PBS. If the two single-photon detectors (D_{A1} and $D_{B1'}$) are simultaneously triggered, the output state is equal to a Bell pair in modes A and B. The optimal success probability to obtain a heralded Bell pair is 1/4 by this method. [21].

Any one of several proposed schemes for generating heralded single photons can be used [22] in the circuit of Fig.1(b). A simple method is to use PDC photons. Naively one may think we need to use 8 PDC pairs, and detection of the 4 idler photons, to generate the four required single photons. In fact, only two PDC events suffice to create a heralded Bell pair. The key idea is that a single higher order PDC emission can generate two single photons simultaneously, as can be seen by studying Eq. (11)) and noting the $|\psi_3^-\rangle$ term in the wavefunction. Looking at this term we see that if one horizontal and one vertical photon are detected in mode B, the photons in mode A are in the state such as $|1,1\rangle_A$. That is, we have picked out the $|\psi_3^-\rangle$ part of the wavefunction. These two photons can be easily used as two of the desired single photons. A separate event from a different PDC would be required to generate the other two single photons. This process is quite feasible - a similar idea was used to demonstrate a path-entangled state in Ref. [23].

3.2. Heralded two-qutrit state from two heralded Bell pairs

A priori one might try and use two pairs of PDC photons in place of the two heralded Bell pairs that we described above. That is, let us assume that two PDC photon pairs (one input in modes A and B, and the other in C and D) are merged into two BSs in Fig. 1(a), and two null detections are successfully achieved. In fact this direct approach produces *no* effective gain in the output state, (over the qutrit state which appears as a higher order effect in the PDC anyway) which means that the output photons are in exactly the same state as the input state. Thus, we focus here on the case that two heralded Bell pairs ($|\psi^2\rangle_{\rm AB}$ and $|\psi^2\rangle_{CD}$) are simultaneously prepared and sent to two BSs with the help of quantum memories, as shown in Fig. 1(a). The initial state is given by

$$|\psi^2\rangle_{AB} \otimes |\psi^2\rangle_{CD}. \tag{17}$$

A 50:50 BS mixes modes A and C, and the other BS does B and D (see Appendix A).

$$(\mathrm{BS}_{AC} \otimes \mathrm{BS}_{BD})|\psi^2\rangle_{AB} \otimes |\psi^2\rangle_{CD}. \tag{18}$$

The output state $|\Psi^{\rm out}\rangle_{ABCD}$ is

$$|\Psi^{\rm out}\rangle_{ABCD} = -\frac{\sqrt{3}}{4}|\psi^3\rangle_{\rm AB} \otimes |0,0\rangle_C \otimes |0,0\rangle_D + |\psi^{\rm rej}\rangle_{ABCD} \tag{19}$$

where the rejected state $|\psi^{\rm rej}\rangle_{ABCD}$ has one or more photons in either mode C or D.

At this stage, an ideal threshold detector can be used for a perfect null detection [24]. The projection operator valued measures (POVM) for the ideal threshold detector in mode C is given by

$$\{\Pi_0 = |0,0\rangle_C\langle 0,0|, \quad \Pi_{>0} = \hat{I} - |0,0\rangle_C\langle 0,0|\}. \tag{20}$$

After two null detections in modes C and D, the final state is equal to

$$\left(\hat{I}_A \otimes \hat{I}_B \otimes |0,0\rangle_C\langle 0,0| \otimes |0,0\rangle_D\langle 0,0|\right)|\Psi^{\rm out}\rangle_{ABCD}. \tag{21}$$

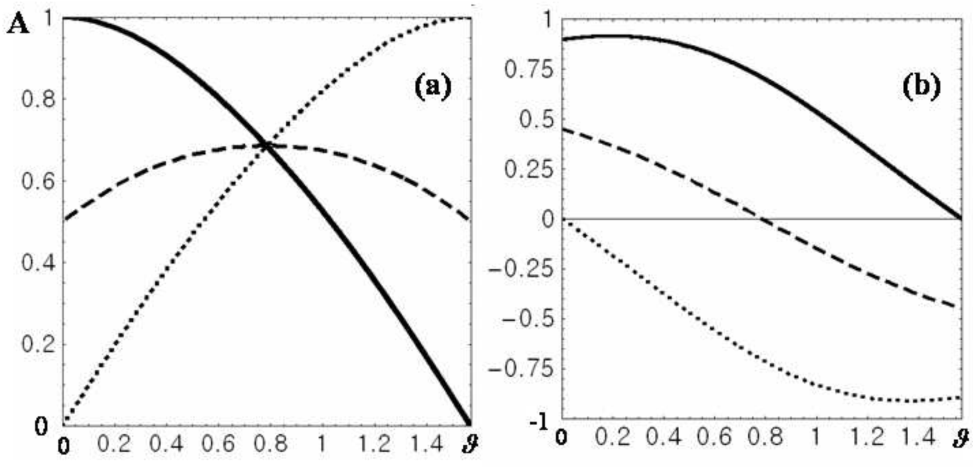

Figure 2. Normalized amplitudes (A) of each terms in Eq. (23) are depicted for (a) $\varphi = 0$ and (b) $\varphi = \pi$. The solid (dotted) line denotes $\cos\vartheta/\sqrt{N'}$ ($\mathrm{e}^{\mathrm{i}\varphi}\sin\vartheta/\sqrt{N'}$), and the dashed line does $(\cos\vartheta + \sin\vartheta)/2\sqrt{N'}$ for (a) and $(\cos\vartheta - \sin\vartheta)/2\sqrt{N'}$ for (b).

For this case, all four incoming photons are merged into modes A and B. Thus, the heralded two-qutrit state with four photons $|\psi^3\rangle_{\mathrm{AB}}$ can be generated with success probability 3/16.

In addition, we are able to produce an unbalanced two-qutrit entangled state through the same circuit. When an unbalanced Bell state is prepared in modes A and B (instead of $|\psi^2\rangle_{\mathrm{AB}}$) such as

$$|\tilde{\psi}^2\rangle_{AB} = \cos\vartheta|1,0\rangle_A|1,0\rangle_B + \mathrm{e}^{\mathrm{i}\varphi}\sin\vartheta|0,1\rangle_A|0,1\rangle_B, \tag{22}$$

the two BSs mix this state with the balanced qutrit state $|\psi^2\rangle_{CD}$. With the null detections in modes C and D, the output state becomes

$$|\tilde{\psi}^3\rangle = \frac{\cos\vartheta}{\sqrt{N'}}|2,0\rangle_A|2,0\rangle_B + \frac{\cos\vartheta + \mathrm{e}^{\mathrm{i}\varphi}\sin\vartheta}{2\sqrt{N'}}|1,1\rangle_A|1,1\rangle_B + \frac{\mathrm{e}^{\mathrm{i}\varphi}\sin\vartheta}{\sqrt{N'}}|0,2\rangle_A|0,2\rangle_B, \tag{23}$$

where N' is a normalization factor. In Fig. 2, the amplitudes of each terms in Eq. (23) are shown for $\varphi = 0$ and $\varphi = \pi$ with respect to angle ϑ. One sees that three amplitudes are the same with $\varphi = 0$ and $\vartheta = \pi/4$ in Fig. 2(a), and this maximally entangled qutrit state is the same as Eq. (7). If $\varphi = \pi$ and $\vartheta = \pi/4$ in Fig. 2(b), the middle term in Eq. (23) disappears and the state equals to

$$\frac{1}{\sqrt{2}}(|2,0\rangle_A|2,0\rangle_B - |0,2\rangle_A|0,2\rangle_B), \tag{24}$$

that is equivalent to a Bell pair with four photons.

4. Applications

A variety of types of generating photonic qutrit entanglement and testing it by Bell inequality [25] violations have been proposed or implemented [26, 27]. For testing the CGLMP inequality in particular, two such schemes are using orbital angular momentum [12] or energy-time [15] entanglement have been experimentally demonstrated. On the theory side a two-photon, two-qutrit state (using multi-modes for the higher dimensions rather than multiple photons) was studied in [5, 14]. In Ref. [13], a PDC photon pair (including higher-order photon emissions) was used to test a type of Clauser-Horne-Shimony-Holt (CHSH) inequality proposed by Gisin and Peres [28]. These schemes all used coincident detection (post-selection). Here we will show how to use our scheme to demonstrate a violation of the CGLMP inequality known as a "tight Bell inequality" for a bipartite qudit state [29]. At the end of this section, we suggest two other possible applications of the heralded qutrit state. The state could be compatible with a scheme for secure quantum communication proposed very recently in Ref. [8]. A non-maximally entangled two-qutrit state could be use of conclusive teleportation probabilistically. We also show how a heralded two-qudit entangled state can be generated by a nested approach of our qutrit scheme.

4.1. CGLMP inequality test

First, the heralded qutrit state is applicable to test the CGLMP inequality [16]. This inequality is a generalized form of the CHSH inequality [30] for a bipartite qudit state, and all the entangled states violate the CGLMP inequality for qudits [31]. The scenario for the CGLMP inequality test involves two parties (A and B). Party A can perform one of two possible measurements (A_1 or A_2) and likewise party B can perform B_1 or B_2 just as for the standard CHSH inequality test. For qutrits, each measurement has three possible outcomes such as

$$A_1, A_2, B_1, B_2 \in 0, 1, 2. \tag{25}$$

The CGLMP inequality consists of four functions of the joint probabilities of the observed outcomes, and each function can take values ± 1 in general (details are given in Section 4.3). However, for example, if three of them are equal to $+1$, the other should be -1 due to the constraint described by local realistic theories. The form of the CGLMP inequality is to show that this particular sum of correlations cannot exceed 2 [16] in a local theory, whereas a value of 2.8729 is achievable by quantum mechanical correlations. This maximum value cannot be achieved using photons in our scheme, because we restrict ourselves to using only linear optical elements for performing the different measurement settings. As such we find the maximum value our scheme can achieve is 2.5295. Perfect photon counting detectors and no photon loss are assumed here. The photon counters can be constructed as an array of the ideal threshold detectors as in Eq. (20) [24].

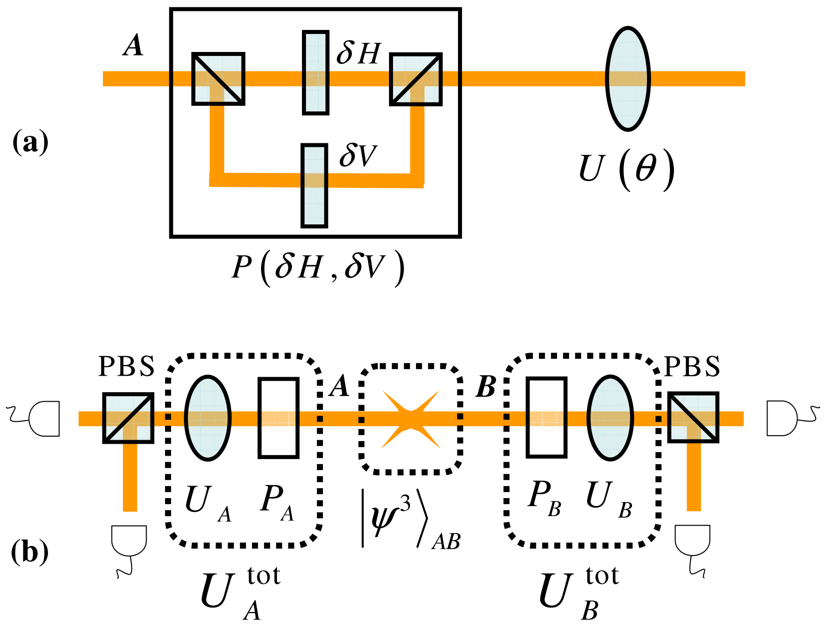

Figure 3. (a) A total single-qutrit operation $U_{\mathrm{A}}^{\mathrm{tot}}(\theta, \delta H, \delta V)$ is decomposed by two operations ($P(\delta H, \delta V)$ and $U(\theta)$) in mode A. Two phase shifters with different angles are located between two PBSs to perform the operation $P(\delta H, \delta V)$. (b) A setup for CGLMP inequality test is depicted.

4.2. local operation on a single qutrit

Let us consider first what kind of single-qutrit operations are available on the bunched photons. Unfortunately, it is impossible to perform an arbitrary single-qutrit operation (even a quantum Fourier transform say) because of the restricted bi-photon operation. The best we can do is a U(2) operation on each photon in the same spatial mode, and this makes a certain single-qutrit operation defined by U^{tot}. As shown in Fig. 3(a), the operator $U_3(\theta)$ is built by a polarization rotator and $P_3(\delta H, \delta V)$ does using two different phase shifters between the two PBSs (see Appendix A). Thus, we here use a simple realization of a single-qutrit operation represented by

$$U^{\mathrm{tot}}(\theta, \delta H, \delta V) = U_3\,(\theta)\;P_3\,(\delta H, \delta V). \tag{26}$$

4.3. CGLMP inequality and four functions of the joint probabilities

We now explain how to test the CGLMP inequality [16] on the two-qutrit state in Eq. (7). According to Ref. [16], the value of the CGLMP inequality for qutrits is given by

$$I_3 = \sum_{i=1}^{4} \mathcal{B}_i \leq 2, \tag{27}$$

where four functions are represented by

$$\mathcal{B}_1 = P(A_1 = B_1) - P(A_1 = B_1 - 1), \tag{28}$$

$$\mathcal{B}_2 = P(B_1 = A_2 + 1) - P(B_1 = A_2), \tag{29}$$

$$\mathcal{B}_3 = P(A_2 = B_2) - P(A_2 = B_2 - 1), \tag{30}$$

$$\mathcal{B}_4 = P(B_2 = A_1) - P(B_2 = A_1 - 1), \tag{31}$$

with a sum of joint probabilities

$$P(A_i = B_j + k) = \sum_{l=0}^{2} P(A_i = l, B_j = k + l \bmod 3). \tag{32}$$

In order to violate the condition $I_3 \leq 2$, we need to choose suitable sets of local operations on both sides ($U^{\mathrm{tot}}_{A_1}$, $U^{\mathrm{tot}}_{A_2}$, $U^{\mathrm{tot}}_{B_1}$, and $U^{\mathrm{tot}}_{B_2}$) and calculate joint probabilities by measuring the qutrit state in modes A and B individually.

To understand what the qutrit operations do, it is easiest to check what is the value for each term $\mathcal{B}_i$ in Eq. (27). As an example, let us choose two local setups A_1 and B_1 such that $U^{\mathrm{tot}}_{A_1}(\theta, 0, 0)$ and $U^{\mathrm{tot}}_{B_1}(\theta, \delta H, 0)$. To obtain the first term of joint probabilities

$$\mathcal{B}_1 = {}_{\mathrm{AB}}\langle \Psi^{11} | O | \Psi^{11} \rangle_{\mathrm{AB}}, \tag{33}$$

the observable is given by

$$O = \left[\sum_{i=0}^{2} |i\rangle_{\mathrm{A}}\langle i| \otimes |i\rangle_{\mathrm{B}}\langle i| \right] - \left[\sum_{i=0}^{2} |i\rangle_{\mathrm{A}}\langle i| \otimes |i + 1 \bmod 3\rangle_{\mathrm{B}}\langle i + 1 \bmod 3| \right], \tag{34}$$

and the state after the operations is equal to

$$|\Psi^{11}\rangle_{\mathrm{AB}} = \left[U^{\mathrm{tot}}_{A_1}(\theta, 0, 0) \otimes U^{\mathrm{tot}}_{B_1}(\theta, \delta H, 0) \right] |\psi^3\rangle_{\mathrm{AB}}. \tag{35}$$

The function of joint probabilities $\mathcal{B}_1$ is limited by

$$-\frac{1}{3} \leq \mathcal{B}_1 \leq 1 \tag{36}$$

and this result shows that the minimum value of $\mathcal{B}_i$ is $-1/3$ [32]. Without the restriction of linear optical elements $\mathcal{B}_i$ can approach to -1.

As a numerical result, the maximum value of I_3 in Eq. (27) ideally reaches $4/(6\sqrt{3} - 9) \approx 2.87293$ [16]. To see how close we can come to this with our restricted operations we numerically examine its violation with 12 parameters in four sets of unitary operators such that $U^{\mathrm{tot}}_{A_1}(\theta_{A_1}, \delta H_{A_1}, \delta V_{A_1})$, $U^{\mathrm{tot}}_{B_1}(\theta_{B_1}, \delta H_{B_1}, \delta V_{B_1})$, $U^{\mathrm{tot}}_{A_2}(\theta_{A_2}, \delta H_{A_2}, \delta V_{A_2})$, and $U^{\mathrm{tot}}_{B_2}(\theta_{B_2}, \delta H_{B_2}, \delta V_{B_2})$. In Fig. 4, the value I_3 is depicted with two variables (x, y) for fixed $\theta = \pi/4$. Its maximum value reaches to 2.5295, and this clearly shows that the maximally entangled qutrit state can violate the CGLMP inequality under the restrictions considered.

4.4. Remarks regarding the inequality test

Space-like separation between the system creating the qutrit state and the measurements performing the Bell inequality tests on the two-qutrit state would be required. That is, the detection of the photons should be undertaken with the null detections done early enough before testing the qutrit state for violating Bell's inequality.

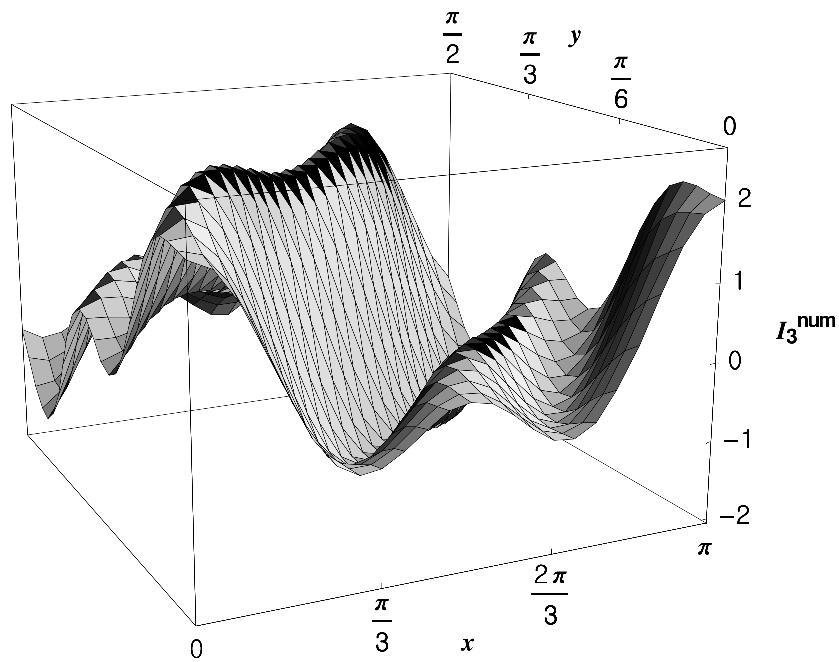

Figure 4. I_3^{num} in Eq. (27) with $U_{A_1}^{\mathrm{tot}}(\pi/4, x, y)$, $U_{B_1}^{\mathrm{tot}}(\pi/4, x, 2y)$, $U_{A_2}^{\mathrm{tot}}(\pi/4, x, 3y)$, and $U_{B_2}^{\mathrm{tot}}(\pi/4, x, 0)$. A maximum value is approximately 2.5295 with $x = y \approx 0.4507$ and $\theta = \pi/4$.

In the quantum channel of our model, we assumed a perfect channel (no photon loss). We note that in the case of PDC the entangled photon pairs are tightly correlated between the frequency and time domains [41]. Because this results in undesired distinguishability of the incoming photons, spectral filters must be positioned in front of detectors. This reduces the rate of successful events and can also be treated as an additional contribution to a finite detection efficiency.

We have also assumed a perfect photon counter is used. Such can be effectively constructed by an array of ideal threshold detectors [24]. Note that imperfect sources and finite detection efficiency would open the same loophole problems for the CGLMP inequality test as they do for the CHSH test. As with those tests, the “no-enhancement” or “fair sampling” assumption would need to be invoked to argue the implausibility of these loopholes. This assumption is basically that the purported hidden variables cannot exploit the finite efficiencies of the sources and detectors by changing themselves from run to run of the test in a way which controls whether they will be detected or not. Alternatively, the Clauser-Horne (CH) inequality [33] can be used when only imperfect sources and detectors are available. This inequality is more robust because “null-detections” (due either to an inefficient detector or a probabilistic source) are taken into account as actual events which enter the correlation functions from which the CH inequality is constructed. The end result is that this inequality is more robust to such noise, and requires less assumptions about fair-sampling. A similar test has been studied for a (different) qutrit system in Ref. [34].

4.5. Other possible applications

The heralded qutrit state has a potential to utilize for quantum cryptography. Very recently, several interesting protocols have been investigated in qutrits for secure quantum key distribution and shown that the qutrit cyptosystem is more secure than qubit-based protocols against some attacks [8]. They are constructed in the spirit of the Bennett and Brassard scheme [35]. In one of the protocols (called three-ray), nine basis vectors (grouped in three basis sets) are randomly chosen by Alice, and a quantum state represented by that vector is sent to Bob. Once Bob received the state and measured it in a certain basis set, he publicly announces which basis set he chose. Finally, Alice confirms whether his choice is correct or not, and both Alice and Bob share a string of raw trits.

One may consider a protocol of quantum key distribution using the entangled two-qutrit state $|\psi^3\rangle_{AB}$. In our system, the three basis sets can be realized by local qutrit operations. Its security could be checked along the same lines as the Ekert protocol in qubits [36]. In analogy with Ekert protocol, once the choice of local operations are matched in both parties, it becomes a raw trit key otherwise they could test a violation of a certain inequality. However, it is an open question which inequality can provide a violation of local hidden theory for this system even though many Bell's inequalities have been studied theoretically for a bipartite system [37, 38]. Therefore, it will be worth studying that which inequality for qutrits could be violated by the qutrit setups.

A two-qutrit state can be used to perform quantum teleportation probabilistically [39]. As an example, an arbitrary single-qutrit state is prepared in mode C proposed in Ref. [40]. Using the circuit shown in Fig. 1(a), the output state in modes A and B can be made as an unbalanced two-qutrit state (see Appendix B)

$$\frac{1}{3}(2|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B + 2|2\rangle_A|2\rangle_B). \tag{37}$$

After a PBS between modes B and C and a polarization rotator $U(\pi/4)$ in both B and C, the unknown state in mode C can be teleported to mode A conclusively by destroying four photons in modes B and C. Four successful detections indicate that the conclusive teleportation is achieved, and the total success probability is 1/9.

The other possible application is that a heralded two-qudit state can be generated by a nested approach of Fig. 1(a). Let us consider that state $|\psi^{d-1}\rangle_{AB}$ is prepared in Fig. 1(a) instead of $|\psi^2\rangle_{AB}$. After two BSs, the case of the null detections on modes C and D repeatedly, the state $|\psi^d\rangle_{AB}$ in Eq.5) is achieved with a success probability

$$P_d = \frac{d(d-1)}{2^{2d-1}}. \tag{38}$$

This circuit can be iteratively used to generate a maximally entangled qudit state. Thus, a heralded two-qudit entangled state can be made from $d-1$ heralded Bell pairs using $2(d-2)$ ideal threshold detectors.

5. Conclusion

We have analyzed a simple scheme for generation of a heralded two-qutrit entangled state in polarized photons. To achieve the qutrit state, when two heralded Bell pairs are mixed by two BSs, two null detections guarantee the generation of the heralded two-qutrit state. We have analyzed the extent to which the qutrit state can be used to demonstrate violation of the CGLMP inequality when measurements are restricted to those achievable with linear optics and photo-detection. We discussed how a source of such states could be compatible with a cryptography scheme proposed in Ref. [8]. We also suggested two possible methods for teleporting an unknown qutrit and creating a heralded two-qudit entangled state by a nested version of our scheme.

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Appendix A. Basic optical tools

A 50:50 BS between modes A and C for a single-photon inputt is given by

$$\mathrm{BS}_{AC} = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}\hat{\mathcal{J}}_{BS}}, \tag{A.1}$$

where

$$\hat{\mathcal{J}}_{BS} = a_H c_H^\dagger + a_H^\dagger c_H + a_V c_V^\dagger + a_V^\dagger c_V, \tag{A.2}$$

The transformation of BS between modes A and C for a single-photon input is given by

$$|1,0\rangle_\mathrm{A} \to \frac{|1,0\rangle_\mathrm{A} + \mathrm{i}|1;0\rangle_C}{\sqrt{2}}, \quad \text{and} \quad |1,0\rangle_C \to \frac{\mathrm{i}|1,0\rangle_\mathrm{A} + |1,0\rangle_C}{\sqrt{2}}, \tag{A.3}$$

where state $|1,0\rangle$ can be replaced to $|0,1\rangle$ for vertically polarized photons.

A polarization rotator with angle θ is given by

$$U(\theta) = \mathrm{e}^{\theta\hat{\mathcal{J}}_R}, \tag{A.4}$$

where

$$\hat{\mathcal{J}}_R = a_V^\dagger a_H - a_H^\dagger a_V. \tag{A.5}$$

For example, if we consider $\left[\hat{I}_A \otimes U_B(\theta)\right]|\Psi\rangle_{AB}$, the unitary operator with angle θ in mode B is represented by a 6×6 matrix form

$$U_B(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_3 \end{pmatrix}, \tag{A.6}$$

where

$$U_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{A.7}$$

$$U_3 = \begin{pmatrix} \cos^2\theta & \sqrt{2}\cos\theta\sin\theta & \sin^2\theta \\ -\sqrt{2}\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta & \sqrt{2}\cos\theta\sin\theta \\ \sin^2\theta & -\sqrt{2}\cos\theta\sin\theta & \cos^2\theta \end{pmatrix}. \tag{A.8}$$

where the basis vectors of Eq. (A.10) are given by $|0,0\rangle$, $|1,0\rangle$, $|0,1\rangle$, $|2,0\rangle$, $|1,1\rangle$, and $|0,2\rangle$. A phase shifter in both polarizations is given by

$$P(\delta H, \delta V) = \mathrm{e}^{\mathrm{i}\delta H \hat{n}_H} \mathrm{e}^{\mathrm{i}\delta V \hat{n}_V}, \tag{A.9}$$

where $\hat{n}_H = a_H^\dagger a_H$, and $\hat{n}_V = a_V^\dagger a_V$ [1, 42]. For example, the phase operator is mode A given by

$$P_A(\delta H, \delta V) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{pmatrix}, \tag{A.10}$$

where

$$P_2 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\delta H} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\delta V} \end{pmatrix}, \tag{A.11}$$

$$P_3 = \begin{pmatrix} \mathrm{e}^{\mathrm{i}2\delta H} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}(\delta H + \delta V)} & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}2\delta V} \end{pmatrix}, \tag{A.12}$$

where the basis vectors are the same as Eq. (A.10).

Appendix B. Building an unbalanced two-qutrit state

We start with the state given by Eq. (24) in modes A and B. When a relative phase is changed by two PBSs and a polarization rotator $U(\pi/2)$, the state becomes

$$|S1\rangle_{AB} = \frac{1}{\sqrt{2}}(|2,0\rangle_A|2,0\rangle_B + |0,2\rangle_A|0,2\rangle_B). \tag{B.1}$$

To achieve the unbalanced state in Eq. (37), an additional entangled state is required in modes C and D such as

$$|S2\rangle_{CD} = \frac{1}{17}(4\sqrt{17}|1,0\rangle_C|1,0\rangle_D + \sqrt{17}|0,1\rangle_C|0,1\rangle_D). \tag{B.2}$$

When the states in Eq. (B.1) and Eq. (B.2) are merged into the circuit shown in Fig. 1(a), the output state becomes

$$(\mathrm{BS}_{AC} \otimes \mathrm{BS}_{BD})|S1\rangle_{AB} \otimes |S2\rangle_{CD}. \tag{B.3}$$

After the detection of a horizontally polarized photon in modes C and D, respectively, the outcome state becomes the state in Eq. (37).

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$$\frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \quad \text{and} \quad \frac{1}{2}\begin{pmatrix} {\rm e}^{{\rm i}2\delta H_0} & \sqrt{2}{\rm e}^{{\rm i}\delta H_0} & 1 \\ -\sqrt{2}{\rm e}^{{\rm i}2\delta H_0} & 0 & \sqrt{2} \\ {\rm e}^{{\rm i}2\delta H_0} & -\sqrt{2}{\rm e}^{{\rm i}2\delta H_0} & 1 \end{pmatrix}, \tag{B.4}$$

respectively, where $\delta H_0 = 2\arccos(1/\sqrt{3}) \approx 1.91063$.

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